

Rényi entropy uncertainty relation for successive projective measurements

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Received: date / Accepted: date

Abstract We investigate the uncertainty principle for two successive projective measurements in terms of Rényi entropy based on a single quantum system. Our results cover a large family of the entropy (including the Shannon entropy) uncertainty relations with a lower optimal bound. We compare our relation with other formulations of the uncertainty principle in a two-spin observables measured on a pure quantum state of qubit. It is shown that the low bound of our uncertainty relation has better tightness.

Keywords Rényi entropy · uncertainty relation · successive projective measurements

1 Introduction

The Heisenberg uncertainty principle (HUP) is one of the well-known fundamental principles in quantum mechanics [1]. It is shown that anyone is not able to specify the values of the non-commuting canonically conjugated variables simultaneously. Later Robertson extended HUP to arbitrary pairs of observables and gave a strict mathematical formulation [2]

$$\Delta X \Delta Y \geq \frac{1}{2} |\langle \Psi | [X, Y] | \Psi \rangle|, \quad (1)$$

where $\Delta X = \sqrt{|\langle \Psi | (X - \langle X \rangle)^2 | \Psi \rangle|}$ represents the variance of the observable X and $[X, Y] = XY - YX$ stands for the commutator. The inequality (1) describes the uncertainty limitations on our ability to simultaneously predict the outcomes of measurements of different observables in quantum theory, even though they can be accurately determined simultaneously in classical theory. One can see that the lower bound is determined by the wavefunction and the commutator of the observables. Subsequently, a large number of researches on the uncertainty relations

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were raised up both in theory [3] and in experiment [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In particular, Deutsch [14], in 1983, presented another uncertainty relation for the conjugate variable observables based on the Shannon entropy [15]. Subsequently, Kraus [16] gave a stronger conjecture of the uncertainty relation, and Maassen and Uffink proved it in a succinct form as [17]

$$H(X) + H(Y) \geq -2 \ln c, \quad (2)$$

where $c = \max_{i,j} |\langle x_i | y_j \rangle|$ quantifies the complementarity of the non-degenerate observables X and Y with $|x_i\rangle, |y_j\rangle$ denoting their corresponding eigenvectors and $H(X)$ is the Shannon entropy of the probability distribution corresponding to the outcomes of the observable X . It is obvious that this lower bound given in Eq. (2) doesn't depend on the state to be measured. Recently, some works that improve the lower bound have been presented by many authors such as Uffink [17], Coles and Piani [18], Łukasz Rudnicki [19] and so on, some works based on different entropies including smooth entropy [20], K -entropy [21], Rényi entropy [22, 23, 24, 25, 26, 27], collision entropy [28, 29], Tsallis entropy *et al.* [30, 31, 32, 33, 34, 35] have been presented, and some works related to different measurements have also been provided for the entropy uncertainty relations [36, 37, 38]. It is worthy of being noted that some interesting results were presented for the uncertainty relation in two-dimensional Hilbert space [39, 40, 41, 42]. They shed new light on our understanding of the uncertainty, even though they could have the limited range of the applications [43, 44, 45, 46]. In addition, most of these mentioned researches are mainly focused on the measurements of two observables which are separately performed on two identical quantum states taken into account. What if the two measurements are successively performed on a quantum state?

In this paper, we investigate the uncertainty relations related to such successive projective measurements. All the quantum operations are also limited in the two-dimensional Hilbert space. We mainly consider two types of the measurement processes of a pair of observables: one is the (usual) successive measurement, that is, the measurement of the second observable is performed on the quantum state generated after the measurement of the first observable with all the information erased, the other is the conditional successive measurement, that is, the measurement of the second observable is performed on the states conditioned on the measurement outcomes of the first observable. By employing the Rényi entropy, we give the explicit uncertainty relations for both processes, that is, the Rényi entropy uncertainty relation (REUR) and the conditional REUR (CREUR). Even though the similar processes were addressed based on the Shannon entropy [41, 42], our results include the previous ones and beyond them in that (1) ours include a large family of the entropy uncertainty relations, since the Rényi entropy is an α -order one-parameter family of entropies [47], the uncertainty relations for different α can compensate for each other; (2) compared with the result in the general spin observables performed on a pure state, it is especially shown that our uncertainty inequalities will be closer to the constant 1 than the corresponding previous ones [42]. The paper is organized as follows. In Sec. II, we provide the Rényi entropy uncertainty relation for successive projective measurements. In Sec. III, we present the conditional Rényi entropy uncertainty relation. In Sec. IV, we take the successive measurements of the general spin observables as an example and compare our uncertainty relations and the previous ones. Finally, we draw the conclusion.

2 Rényi entropy uncertainty relation for successive measurement

To begin with, let's give a brief introduction of the Rényi entropy which was given by Rényi in 1961 [47]. For the probability distribution $\{p_i\}$, $0 \leq p_i \leq 1$ and $\sum_{i=1}^N p_i = 1$, the Rényi entropy is defined by

$$R_\alpha(\{p_i\}) = \frac{1}{1-\alpha} \ln \left(\sum_{i=1}^N p_i^\alpha \right), \quad (3)$$

with $\alpha \geq 0$ and $\alpha \neq 1$. The Rényi entropy covers a large family of entropies with different indices α taken into account. (1) If $\alpha = 0$, it corresponds to a trivial result, the max-entropy $\ln N$; (2) If $\alpha \rightarrow 1$, R_α approaches the Shannon entropy $R_1 = -\sum_{i=1}^N p_i \ln(p_i)$; (3) If $\alpha = 2$, the Rényi entropy becomes $R_2 = -\ln \left(\sum_{i=1}^N p_i^2 \right)$ which is the collision entropy; (4) If $\alpha \rightarrow \infty$, R_α is known as min-entropy, $R_\infty = -\ln \max_i [p_i]$.

In order to give our main results, we would like to turn to the Bloch representation. Considering the non-degenerate two-dimensional observables P and Q with the eigenvectors (orthogonal projectors) denoted by \hat{P}_1 (\hat{Q}_1) and \hat{P}_2 (\hat{Q}_2), they can be given in the Bloch representation by

$$\begin{cases} \hat{P}_1 = \frac{1}{2}(1 + \vec{p} \cdot \vec{\sigma}) \\ \hat{P}_2 = \frac{1}{2}(1 - \vec{p} \cdot \vec{\sigma}) \end{cases}, \begin{cases} \hat{Q}_1 = \frac{1}{2}(1 + \vec{q} \cdot \vec{\sigma}) \\ \hat{Q}_2 = \frac{1}{2}(1 - \vec{q} \cdot \vec{\sigma}) \end{cases}, \quad (4)$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the standard Pauli matrices and $|\vec{p}| = |\vec{q}| = 1$. Throughout this paper, we only consider the non-degenerate observables P and Q . The degenerate case can be easily derived based on our results. Similarly, the quantum state of a qubit can also be written as

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}), \quad (5)$$

with $|\vec{r}| \leq 1$. Note that the quantum state denotes a pure state if $|\vec{r}| = 1$, otherwise it represents a mixed state. With these in mind, our results can be given as follows.

Theorem.1. For the successive measurements of the two observables P and Q operate on any quantum state ρ , the REUR is given by

$$R_\alpha(P) + R_\alpha^{\varepsilon(\rho)}(Q) \geq R_\alpha(\rho) + R_\alpha(Q_\pm), \quad (6)$$

with $Q_\pm = \frac{1 \pm (\vec{p} \cdot \vec{q}) |\vec{r}|}{2}$. In particular, the superscript $\varepsilon(\rho)$ denotes the final state generated by the measurement P with all the information about the measurement outcomes erased.

Proof. Let the measured quantum state be ρ , then the probability of the measurement outcomes for the observable P reads

$$P_1 = \text{tr} \hat{P}_1 \rho = \frac{1}{2}(1 + \vec{p} \cdot \vec{r}), \quad (7)$$

$$P_2 = \text{tr} \hat{P}_2 \rho = \frac{1}{2}(1 - \vec{p} \cdot \vec{r}). \quad (8)$$

So the post-measured state with all the information of the measurement outcomes erased is given by

$$\varepsilon(\rho) = P_1 \hat{P}_1 + P_2 \hat{P}_2 = \frac{1}{2} \left(1 + \vec{k} \cdot \vec{\sigma} \right), \quad (9)$$

with $\vec{k} = (\vec{p} \cdot \vec{r}) \vec{p}$. Next we perform the observable Q on the state $\varepsilon(\rho)$ and get the probability distribution as

$$Q_1 = \text{tr} \hat{Q}_1 \varepsilon(\rho) = \frac{1}{2} (1 + \vec{k} \cdot \vec{q}), \quad (10)$$

$$Q_2 = \text{tr} \hat{Q}_2 \varepsilon(\rho) = \frac{1}{2} (1 - \vec{k} \cdot \vec{q}). \quad (11)$$

Substitute Eqs. (7) and (8) and Eqs. (10) and (11) into Eq. (3), one will arrive at

$$\begin{aligned} R_\alpha(P) + R_\alpha^{\varepsilon(\rho)}(Q) &= \frac{1}{1-\alpha} \ln \sum P_i^\alpha + \frac{1}{1-\alpha} \ln \sum Q_i^\alpha \\ &= \frac{1}{1-\alpha} \ln \sum P'_\pm{}^\alpha \sum Q'_\pm{}^\alpha, \end{aligned} \quad (12)$$

with

$$P'_\pm = \frac{1 \pm |\vec{r}| \cos \theta}{2}, Q'_\pm = \frac{1 \pm m |\vec{r}| \cos \theta}{2}, \quad (13)$$

where $m = \vec{p} \cdot \vec{q}$ and $\vec{p} \cdot \vec{r} = |\vec{r}| \cos \theta$, $\theta \in [0, \pi]$ is the angle between the Bloch vectors of the observable P and the initial state ρ . Next, we will derive the lower bound of Eq. (12) suited to all the states by two cases.

(a) If $m \neq 0$, we assume that $f(x) = \frac{1}{1-\alpha} \ln \left[\left(\frac{1+mx}{2} \right)^\alpha + \left(\frac{1-mx}{2} \right)^\alpha \right]$. The derivative of x on $f(x)$ shows

$$\frac{\partial f(x)}{\partial x} = \frac{1}{1-\alpha} \frac{\alpha m [(1+mx)^{\alpha-1} - (1-mx)^{\alpha-1}]}{(1+mx)^\alpha + (1-mx)^\alpha}. \quad (14)$$

It is obvious that $\frac{\partial f(x)}{\partial x} = 0$ implies $x = 0$. One can easily find that for any $m \neq 0$, $f(x)$ monotonically decreases for $x \in [0, 1]$ and monotonically increases for $x \in [-1, 0]$. So the minimum of $f(x)$ could be reached at ± 1 . In addition, we have $f(1) = f(-1)$. Thus, we can find that the lower bound of Eq. (12) can be given by

$$R_\alpha(P) + R_\alpha^{\varepsilon(\rho)}(Q) \geq R_\alpha(\rho) + R_\alpha(Q_\pm), \quad (15)$$

with $Q_\pm = \frac{1 \pm (\vec{p} \cdot \vec{q}) |\vec{r}|}{2}$. The inequality will be saturated if $\cos \theta = \pm 1$, that is, the Bloch vectors of the observable P and the initial state ρ are parallel.

(b) If $m = 0$, Eq. (12) will become

$$R_\alpha(P) + R_\alpha^{\varepsilon(\rho)}(Q) = \frac{1}{1-\alpha} \ln \left(\sum P'_\pm{}^\alpha \frac{1}{2^{\alpha-1}} \right). \quad (16)$$

Based on the properties of $f(x)$, the lower bound can be given by

$$R_\alpha(P) + R_\alpha^{\varepsilon(\rho)}(Q) \geq R_\alpha(\rho) + 1, \quad (17)$$

The "=" holds under the same condition as the case (a). Eqs. (15) and (17) complete the proof. \blacksquare

It is obvious that the REUR in above theorem partially depends on the measured state, since it includes the length of the Bloch vector $|\vec{r}|$. In this sense, it is much like the HUP given by Eq. (1). In addition, as is known to all, the Rényi entropy is concave for $\alpha \leq 1$, but it is neither convex nor concave for $\alpha > 1$. So most relevant works based on the Rényi entropy for $\alpha \leq 1$ are only considered

for the pure states, but they can be naturally extended to mixed states due to the concave property. However, one of the merits of our current result is suited to both pure and mixed states for all α , which can be seen from that our proof is directly based on the mixed states. What's more, one can find that the uncertainty relations depend on the measurements and the states. Based on the above proof, we would like to emphasize that given a pair of measurements, the states with the Bloch vector parallel with the Bloch vector of P will achieve the minimal uncertainty for any α . Finally, it is shown that different α corresponds to different entropies. It is worth to emphasize that $\alpha = 0$ corresponds to the max-entropy that equals to $\ln M$, where M is the number of strictly positive probabilities. In more dimensions, however, a consideration of the max-entropy will be non-trivial. In the two-dimensional case, therefore, nonzero max-entropies are all $\ln 2$. So the low bound of Rényi entropy uncertainty relation for successive measurement will become the trivial result $2\ln 2$. $\alpha \rightarrow 1$ and $\alpha \rightarrow \infty$ correspond to the Shannon entropy and the min-entropy, respectively. For these two cases, we can obtain the following corollary from Theorem. 1.

Corollary.1. The uncertainty relations of the successive projective measurements based on $\alpha = 1$ and $\alpha \rightarrow \infty$ will be given by

$$H(P) + H^{\varepsilon(\rho)}(Q) \geq H(\rho) + H(Q_{\pm}), \quad (18)$$

$$R_{\infty}(P) + R_{\infty}^{\varepsilon(\rho)}(Q) \geq -\ln \left(\max_i [P_i] \max_j [Q_j] \right), \quad (19)$$

with $P_{\pm} = \frac{1 \pm |\vec{r}|}{2}$, $Q_{\pm} = \frac{1 \pm (\vec{p} \cdot \vec{q}) |\vec{r}|}{2}$ and $\max_i [P_i]$ denoting the maximum of P_i .

Proof. Comparing Eq. (3) and the right-hand side of Eq. (6), one can easily prove this corollary. ■

2.1 Conditional Rényi entropy

At first, we briefly introduce the definition of the conditional Rényi entropy. It is defined for $\alpha \geq 0$ with $\alpha \neq 1$ by [48, 49],

$$R_{\alpha}(Q|P) = \sum P_i R_{\alpha}(Q|P = p_i) = \frac{1}{1-\alpha} \sum P_i \ln P_{\hat{Q}_j|\hat{P}_i}^{\alpha}, \quad (20)$$

where P_i is the probability corresponding to the projector \hat{P}_i and $P_{\hat{Q}_j|\hat{P}_i}$ is the probability corresponding to the projector \hat{Q}_j conditioned on the projector \hat{P}_i .

Considering the successive measurements of the observables P and Q in the Bloch representation, one will obtain the conditional Rényi entropy as follows.

Theorem.2. The conditional Rényi entropy of two successive measurements P and Q on any state ρ is given by

$$R_{\alpha}(Q|P) = \frac{1}{1-\alpha} \ln \sum K_{\pm}^{\alpha}, \quad (21)$$

with $K_{\pm} = \left(\frac{1 \pm \vec{p} \cdot \vec{q}}{2} \right)$.

Proof. For the quantum state ρ , we first perform the measurement of P with the final state (projector) given by \hat{P}_i and the corresponding probability given by

Eqs. (7) and (8), and then perform the measurement of Q on \hat{P}_i . So the conditional probability distribution is given by

$$P_{\hat{Q}_1|\hat{P}_1} = \text{tr} \hat{Q}_1 \rho_{(1)}^P = \frac{1}{2}(1 + \vec{q} \cdot \vec{p}), \quad (22)$$

$$P_{\hat{Q}_2|\hat{P}_1} = \text{tr} \hat{Q}_2 \rho_{(1)}^P = \frac{1}{2}(1 - \vec{q} \cdot \vec{p}), \quad (23)$$

$$P_{\hat{Q}_1|\hat{P}_2} = \text{tr} \hat{Q}_1 \rho_{(2)}^P = \frac{1}{2}(1 - \vec{q} \cdot \vec{p}), \quad (24)$$

$$P_{\hat{Q}_2|\hat{P}_2} = \text{tr} \hat{Q}_2 \rho_{(2)}^P = \frac{1}{2}(1 + \vec{q} \cdot \vec{p}). \quad (25)$$

Insert Eqs. (22-25) into Eq. (20), one will arrive at Eq. (21), which ends the proof. ■

Since the uncertainty relation describes the constraint on the measurement outcomes of two observables, the conditional Rényi entropy directly implies such an uncertainty relation which is named as the conditional Rényi entropy uncertainty relation (CREUR). However, we would like to emphasize that the CREUR corresponds to a different measurement procedure from that of REUR. In addition, one can find that the conditional Rényi entropy given by Eq. (21) doesn't depend on the measured state.

2.2 Conditional Rényi entropy as the lower bound

As mentioned above, the REUR depends on the measured state. However, via a further optimization on $|\vec{\tau}|$, we can obtain another REUR independent of the measured state. This is given by what follows.

Theorem. 3. For the successive measurements of the two observables P and Q operated on a quantum state ρ , the REUR independent of the measured state is given by

$$R_\alpha(P) + R_\alpha^{\varepsilon(\rho)}(Q) \geq R_\alpha(Q|P). \quad (26)$$

The equality is saturated if the initial quantum state is the eigenvector of P .

Proof. Based on the properties of $f(x)$ given by the proof of Theorem.1, one can further obtain the lower bound given by Eq. (26) with the lower bound reached when $|\vec{\tau}| = 1$, which means the pure quantum state is the eigenvector of P . ■

Corollary.2. Eq. (26) for the Shannon entropy uncertainty relation and the min-entropy uncertainty relation are, respectively, reduced to

$$H(P) + H^{\varepsilon(\rho)}(Q) \geq H(Q|P), \quad (27)$$

$$R_\infty(P) + R_\infty^{\varepsilon(\rho)}(Q) \geq -\ln \left(\max_i [K_i] \right), \quad (28)$$

with i denoting \pm .

Proof. The proof is similar to that of Corollary. 1. ■

3 Comparison between various uncertainty relations

In this section, we take the general spin observables performed on a pure state as example to compare our mentioned uncertainty relations. In the Bloch representation, the two considered spin observables X and Y with the intersection angle 2ϕ can always be arranged in the $x - y$ plane. Thus, they can be written as

$$X = \cos \phi \sigma_x + \sin \phi \sigma_y, \quad (29)$$

$$Y = \sin \phi \sigma_x + \cos \phi \sigma_y. \quad (30)$$

where σ_x and σ_y are the Pauli matrices. Similarly, a pure state in this representation can be given by $\rho = |\psi\rangle\langle\psi|$, where

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle, \quad (31)$$

with the azimuthal angle $0 \leq \varphi \leq 2\pi$ and the polar angle $0 \leq \theta \leq \pi$. If the initial state ρ is measured by the observable \hat{X} , the probability for different measurement outcomes reads

$$P_{\pm}^x = \frac{1}{2} [1 \pm \cos(\phi - \varphi) \sin \theta], \quad (32)$$

with \pm distinguishing the different measurement outcomes, and the post-measured state reads

$$\varepsilon(\rho) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} e^{-i\phi} \cos(\phi - \varphi) \sin \theta \\ \frac{1}{2} e^{i\phi} \cos(\phi - \varphi) \sin \theta & \frac{1}{2} \end{pmatrix}. \quad (33)$$

The measurement of the observable Y will generate the probability distribution as

$$P_{\pm}^{\varepsilon(\rho)y} = \frac{1}{2} [1 \pm \cos(\phi - \varphi) \sin \theta \sin 2\phi]. \quad (34)$$

Analogously, one can find that the conditional probability distribution can be given by

$$P_{(y+|x+)} = P_{(y-|x-)} = \frac{1}{2} (1 + \sin 2\phi), \quad (35)$$

$$P_{(y-|x+)} = P_{(y+|x-)} = \frac{1}{2} (1 - \sin 2\phi), \quad (36)$$

where $x\pm$ ($y\pm$) denotes the measurement outcomes \pm of the observable X (Y). Therefore, the REUR and the CREUR can be easily obtained

$$R_{\alpha}(X) + R_{\alpha}^{\varepsilon(\rho)}(Y) \geq R_{\alpha}(X|Y), \quad (37)$$

$$R_{\alpha}(X|Y) = \frac{1}{1-\alpha} \ln \sum P_{\pm}^{\alpha}, \quad (38)$$

with $P_{\pm}^{\alpha} = \frac{1 \pm \sin 2\phi}{2}$. Note that different α corresponds to the different entropy uncertainty relation. In order to show the tightness of the inequality of the uncertainty relations, we would like to calculate the Shannon entropy uncertainty

relation previously (SEURp) and the conditional Shannon entropy uncertainty relation preciously (CSEURp) given by [42] as comparisons. The SEURp and the CSEURp for the current example can be calculated as

$$H(X) + H^{\varepsilon(\rho)}(Y) \geq -2 \ln c, \quad (39)$$

$$H(Y|X) \geq -2 \ln c, \quad (40)$$

with $c = \sqrt{\frac{1+\sin 2\phi}{2}}$ for $\phi \in [0, \frac{\pi}{2}]$.

From Eqs. (37) and (38), one can obtain the Shannon entropy uncertainty relation (SEUR) by setting $\alpha \rightarrow 1$. However, comparing Eqs. (37) and (38) and Eqs. (39) and (40), one can easily find that the tightness of REUR (CREUR) for $\alpha \rightarrow 1$ is not worse than that of SEURp (CSEURp), since Eq. (38) is an equality and Eq. (40) is an inequality. This can also be explicitly found from our latter illustrations. To reveal more in what our uncertainty relation covers, inspired by the methods of Ref. [42] we would like to reformulate Eqs. (37-40), respectively, by the following form,

$$\frac{R_\alpha(X) + R_\alpha^{\varepsilon(\rho)}(Y)}{\frac{1}{1-\alpha} \ln \left[\left(\frac{1+\sin 2\phi}{2} \right)^\alpha + \left(\frac{1-\sin 2\phi}{2} \right)^\alpha \right]} \geq 1, \quad (41)$$

$$\frac{R_\alpha(X|Y)}{\frac{1}{1-\alpha} \ln \left[\left(\frac{1+\sin 2\phi}{2} \right)^\alpha + \left(\frac{1-\sin 2\phi}{2} \right)^\alpha \right]} = 1, \quad (42)$$

$$\frac{H(X) + H^{\varepsilon(\rho)}(Y)}{-\ln \frac{1+\sin 2\phi}{2}} \geq 1, \quad (43)$$

$$\frac{H(Y|X)}{-\ln \frac{1+\sin 2\phi}{2}} \geq 1. \quad (44)$$

Thus, the more the left-hand side approaches 1, the tighter the inequality is. In Fig. 1, we compare REUR and SEURp with different α in Eqs. (41) and (43). Here we set $\phi = 0$, that is, the Bloch vectors of the observables X and Y are along the x, y axes. The left-hand sides (signed by ‘uncertainty’) of Eqs. (41) and (43) are plotted versus φ for fixed angles $\theta = 0, \pi/4, 5\pi/9, \pi/2$ and different $\alpha = 1/2, 1, 2, \infty$. It is clear that the SEURp well coincides with the REUR for $\alpha \rightarrow 1$, i.e., SEUR. In particular, one can see that in the current case, for the larger α or θ the REUR will be close to the constant 1, which shows the better tightness. When the polar angle θ goes to $\pi/2$, the two endpoints of the SEURp and the REUR for all α will reach the optimal solution. In Fig. 2, fixing $\theta = \pi/4$, we plot the ‘uncertainty’ versus φ for different $\phi = 0, \pi/7, \pi/3, \pi/2$ and $\alpha = 1/2, 1, 2, \infty$. When ϕ takes $0, \pi/2$, the SEURp is also consistent with the SEUR, which is analogous to that in Fig. 1, but in Fig. 2 b and Fig. 2 c, the SEUR (Eq. 41) is different from the SEURp (Eq. 43), while the SEUR is closer to the optimal regime. In addition, in Fig. 2 (a) and 2 (d), the larger α corresponds to better tightness, but in Fig. 2 (b) and 2 (c), the case is inverse. In Fig. 3, we mainly show that how the CSEURp and the CREUR depend on φ for different ϕ . Here we plot the cases for $\phi = 0 (\pi/2), \pi/15 (13\pi/30), \pi/10 (2\pi/5), \pi/5 (3\pi/10)$. One can easily find that the CREUR given by Eq. (42) for all α and the CSEURp for $\phi = 0(\pi/2)$ is superposable at the bound line 1. But the CSEURp is separated by different ϕ . In fact, it can be found that the lower bound of the CSEURp grows

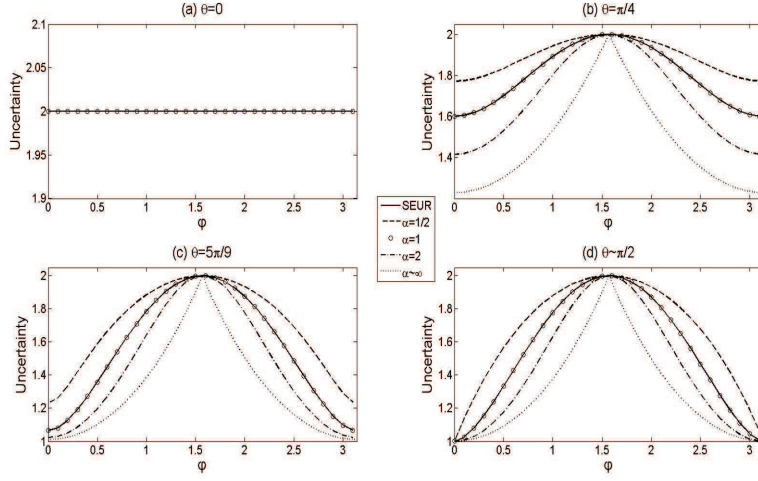


Fig. 1 The SEURp and the REUR vs. ϕ for different θ . We set $\phi = 0$. In each figure, from the top to the bottom, the lines correspond to α takes $1/2, 1, 2, \infty$, respectively.

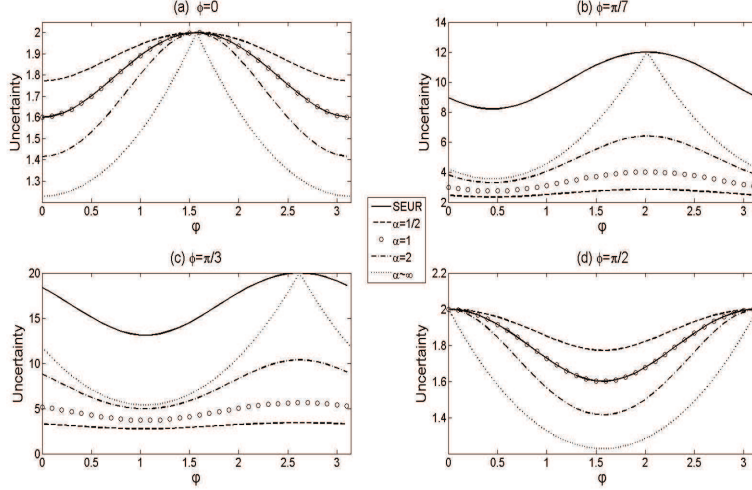


Fig. 2 The SEURp and the REUR vs. ϕ for different ϕ . We set $\theta = \pi/4$. From the top to the bottom in figure (a) and (d), the lines correspond to $\alpha = 1/2, 1, 2, \infty$, respectively, but in figure (b) and (c) they are converse.

up with the increase in ϕ , but it tends to infinity if $\phi \rightarrow \pi/4$ and then decreases. The periodicity for ϕ is $\pi/4$. Thus, it explicitly shows that the tightness of the CREUR is better than that of the CSEURp.

Since Ref. [4] has improved the bound of Eq. (2), we would like to briefly compare the tightness of our uncertainty relation for $\alpha \rightarrow 1$ and the improved entropy uncertainty. In the current case, the improved version of Eq. (2) given in

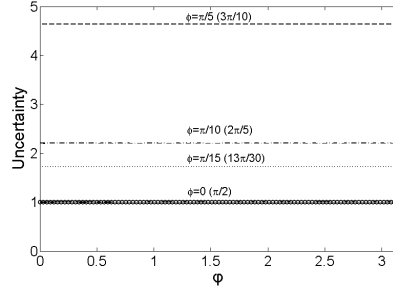


Fig. 3 The CSEURp and the CREUR vs. φ for different ϕ . We set $\theta = \pi/4$. The CREUR (circle) for all α and CSEURp (line) for $\phi = 0$ are superposable at the bound line 1, but the CSEURp is separated by different ϕ .

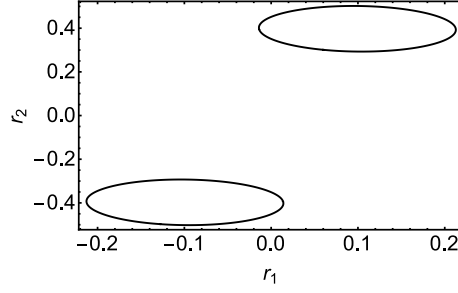


Fig. 4 The inner ‘ellipse’ regions show that our bound for $\alpha \rightarrow 1$ is tighter than Eq. (45)

Ref. [4] reads

$$H^{\mathcal{E}(\rho)}(P) + H^{\mathcal{E}(\rho)}(Q) \geq H(\mathcal{E}(\rho)) - 2 \ln c. \quad (45)$$

One could have noted that for $\vec{p} \perp \vec{q}$, or the pure ρ (not eigenstate of \hat{P}), our bound for $\alpha \rightarrow 1$ is not better than Eq. (45). However, the uncertainty relations depend on both the measurements and the state. In particular, the entropy is defined by the logarithm function; hence, it is hard to give an analytic expression to show in which cases our bound is better. So in the following, we will give a simple example to show that there exist some cases for which our bound is tighter than Eq. (45) indeed. Here we let $p_1 = 0.1, p_2 = 0.4, p_3 = \sqrt{1 - p_1^2 - p_2^2}, q_1 = 0.15, q_2 = 0.5, q_3 = \sqrt{1 - q_1^2 - q_2^2}$. Through a simple numerical procedure, one can find that the ‘ellipse’ regions in Fig. 4 show that our bound is tighter than Eq. (45).

4 Discussions and Conclusion

We have proposed the Rényi entropy uncertainty relation and the conditional Rényi entropy uncertainty relation for successive non-degenerate measurements of arbitrary pairs of two-dimensional observables. Our results cover a variety of the entropy uncertainty relations including the previous Shannon entropy uncertainty relations (SEURp and CSEURp). Through the comparisons, for some particular α , our uncertainty inequalities are closed to the constant 1, that is, it has better tightness than the SEURp and the CSEURp, which supports the necessity of

studying a large family of the uncertainty relations instead of a single one. In addition, we would like to say that Theorem. 1 provides a subtle description of the uncertainty even though it is state dependent. Finally, one can see that the Rényi entropy uncertainty relation and the conditional Rényi entropy uncertainty relation for successive non-degenerate measurements of arbitrary pairs of two-dimensional observables are only sufficient for qubit systems. We look forward to the latter progress on the N-level systems.

5 Acknowledgement

This work was supported by the National Natural Science Foundation of China, under Grants numbers 11375036 and 11175033, and the Xinghai Scholar Cultivation Plan.

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